Applications - Chapitre 1

Etudier la mécanique

EPFL

A.1.1 Dérivées de fonctions

- Binôme de Newton :

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!}\,a^{n-k}b^k = a^n + n\,a^{n-1}b + \sum_{k=2}^{n} \frac{n!}{(n-k)!\,k!}\,a^{n-k}b^k$$

1. $f(x) = x^n$ où $x \in \mathbb{R}$

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^n} + n\,x^{n-1}\Delta x + \cancel{\sum_{k=2}^{n} \frac{n!}{(n-k)!\,k!}\,x^{n-k}\Delta x^k} - \cancel{x^n}}{\Delta x}$$

$$= n\,x^{n-1}$$

$$\frac{d(x^n)}{dx} = n\,x^{n-1} \tag{A.1.1}$$

- Formule de trigonométrie :

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

2. $f(x) = \sin(x)$ où $x \in \mathbb{R}$

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x}$$

$$= \underbrace{\left(\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x}\right)}_{=0} \sin(x) + \underbrace{\left(\lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}\right)}_{=1} \cos(x)$$

$$= \cos(x)$$

$$\frac{d\sin(x)}{dx} = \cos(x) \qquad (A.1.2)$$

- Formule de trigonométrie :

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

3. $f(x) = \cos(x)$ où $x \in \mathbb{R}$

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos(x)\cos(\Delta x) - \sin(x)\sin(\Delta x) - \cos(x)}{\Delta x}$$

$$= \underbrace{\left(\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x}\right)}_{=0} \cos(x) - \underbrace{\left(\lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}\right)}_{=1} \sin(x)$$

$$= -\sin(x)$$

$$\boxed{\frac{d\cos(x)}{dx} = -\sin(x) \qquad (A.1.3)}$$

- Identité exponentielle :

$$\exp(a+b) = \exp(a)\,\exp(b)$$

4. $f(x) = \exp(x) \quad$ où $\quad x \in \mathbb{R}$

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\exp(x+\Delta x) - \exp(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\exp(x)\,\exp(\Delta x) - \exp(x)}{\Delta x}$$

$$= \underbrace{\left(\lim_{\Delta x \to 0} \frac{\exp(\Delta x) - 1}{\Delta x}\right)}_{=1} \exp(x) = \exp(x)$$

$$\frac{d\exp(x)}{dx} = \exp(x) \qquad (A.1.4)$$

- Identité logarithmique :

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

5. $f(x) = \ln(x)$ où $x \in \mathbb{R}_+^*$

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln\left(\frac{x+\Delta x}{x}\right)$$

$$= \underbrace{\left(\lim_{\Delta x/x \to 0} \frac{x}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right)\right)}_{=1} \frac{1}{x} = \frac{1}{x}$$

$$\frac{d\ln(x)}{dx} = \frac{1}{x} \qquad (A.1.5)$$

1. $g(x) = \exp(f(x))$ où $x \in \mathbb{R}$ et $f : \mathbb{R} \to \mathbb{R}$ de classe C^1

$$\frac{dg(x)}{dx} = \frac{d\exp(f(x))}{dx} = \frac{d\exp(f(x))}{df(x)}\,\frac{df(x)}{dx}$$

$$= \left(\lim_{\Delta f(x)\to 0} \frac{\exp(f(x) + \Delta f(x)) - \exp(f(x))}{\Delta f(x)}\right) \frac{df(x)}{dx}$$

$$= \underbrace{\left(\lim_{\Delta f(x)\to 0} \frac{\exp(\Delta f(x)) - 1}{\Delta f(x)}\right)}_{=1} \exp(f(x))\,\frac{df(x)}{dx}$$

$$= \frac{df(x)}{dx}\,\exp(f(x))$$

$$\frac{d\exp(f(x))}{dx} = \frac{df(x)}{dx}\,\exp(f(x)) \qquad (A.1.6)$$

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2. $g(x) = \ln(f(x))$ où $x \in \mathbb{R}$ et $f : \mathbb{R} \to \mathbb{R}_+^*$ de classe C^1

$$
\begin{aligned}
\frac{dg(x)}{dx} = \frac{d\ln(f(x))}{dx} &= \frac{d\ln(f(x))}{df(x)}\,\frac{df(x)}{dx} \\
&= \left(\lim_{\Delta f(x)\to 0} \frac{\ln(f(x)+\Delta f(x)) - \ln(f(x))}{\Delta f(x)}\right)\frac{df(x)}{dx} \\
&= \left(\lim_{\Delta f(x)\to 0} \frac{1}{\Delta f(x)}\ln\left(\frac{f(x)+\Delta f(x)}{f(x)}\right)\right)\frac{df(x)}{dx} \\
&= \underbrace{\left(\lim_{\Delta f(x)/f(x)\to 0} \frac{f(x)}{\Delta f(x)}\ln\left(1+\frac{\Delta f(x)}{f(x)}\right)\right)}_{=1}\frac{df(x)}{dx}\,\frac{1}{f(x)} \\
&= \frac{df(x)}{dx}\,\frac{1}{f(x)}
\end{aligned}
$$

$$
\frac{d\ln(f(x))}{dx} = \frac{df(x)}{dx}\,\frac{1}{f(x)} \tag{A.1.7}
$$

$$\frac{d\ln\left(f\left(x\right)\right)}{dx} = \frac{df\left(x\right)}{dx}\,\frac{1}{f\left(x\right)} \tag{A.1.7}$$

- Intégration de l'équation $(A.1.7)$ sur $x \in [a, b]$:

$$\int_a^b \left(\frac{d\ln\left(f\left(x\right)\right)}{dx}\right) dx = \int_{f(a)}^{f(b)} \frac{df'\left(x\right)}{f'\left(x\right)}$$

où

$$\int_a^b \left(\frac{d\ln\left(f\left(x\right)\right)}{dx}\right) dx = \ln\left(f\left(x\right)\right)\Big|_{x=a}^{x=b} = \ln\left(f\left(b\right)\right) - \ln\left(f\left(a\right)\right) = \ln\left(\frac{f\left(b\right)}{f\left(a\right)}\right)$$

ainsi

$$\boxed{\int_{f(a)}^{f(b)} \frac{df'\left(x\right)}{f'\left(x\right)} = \ln\left(\frac{f\left(b\right)}{f\left(a\right)}\right)} \tag{A.1.8}$$

- On considère un pendule de longueur $\ell =$ cste suspendu à l'origine O d'un système d'axes cartésiens Oxy.

1. Coordonnée de position (abscisse):

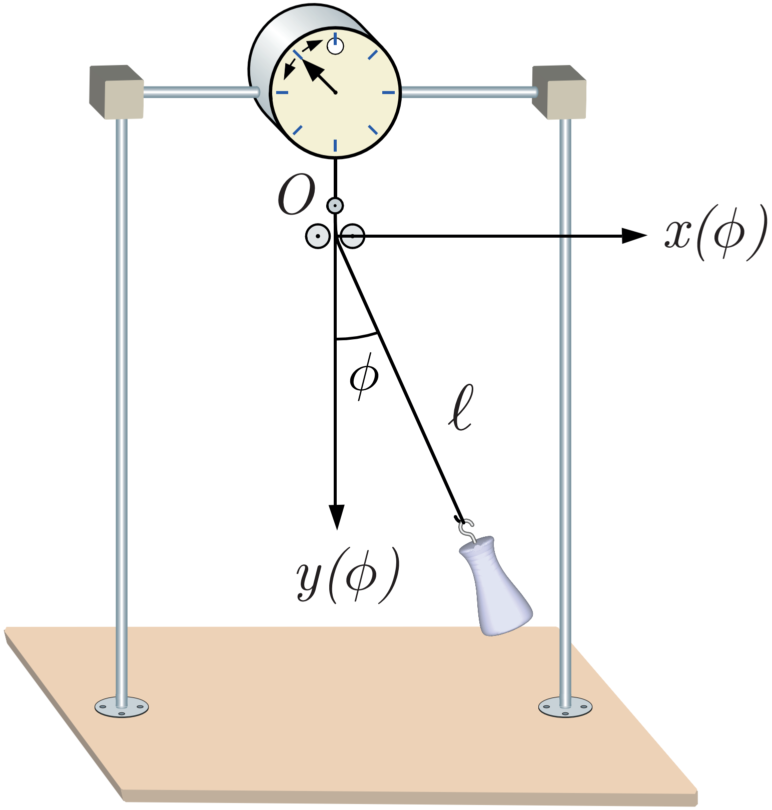
$$x\left(\phi\right) = \ell \sin\phi$$

- Développement limité de $x\left(\phi\right)$ au 1^{er} ordre en ϕ autour de $\phi = 0$ (i.e. $\phi \ll 1$) :

$$x\left(\phi\right) \simeq x\left(0\right) + \frac{dx}{d\phi}\left(0\right)\phi$$

$$= \ell \underbrace{\sin\left(0\right)}_{=0} + \ell \underbrace{\cos\left(0\right)}_{=1}\phi = \ell\,\phi$$

$$x\left(\phi\right) \simeq \ell\,\phi \quad \text{si} \quad \phi \ll 1 \qquad (A.1.9)$$



2. Coordonnée de position (ordonnée):

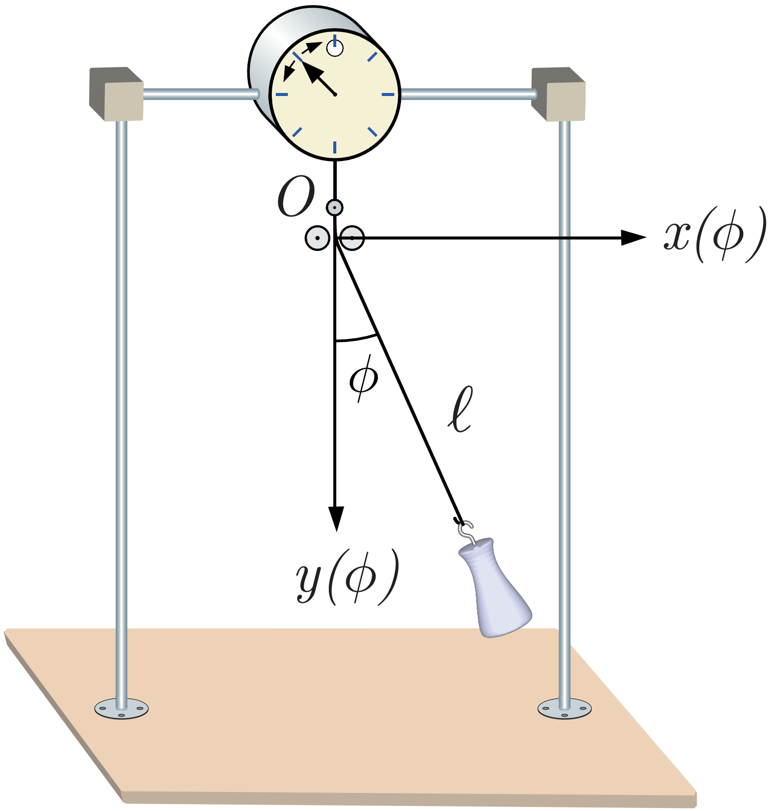
$$y\left(\phi\right) = \ell \cos\phi$$

- Développement limité de $y\left(\phi\right)$ au 1^{er} ordre en ϕ autour de $\phi = 0$ (i.e. $\phi \ll 1$) :

$$y\left(\phi\right) \simeq y\left(0\right) + \frac{dy}{d\phi}\left(0\right)\phi$$

$$= \ell \underbrace{\cos\left(0\right)}_{=1} - \ell \underbrace{\sin\left(0\right)}_{=0} \phi = \ell$$

$$y\left(\phi\right) \simeq \ell \quad \text{si} \quad \phi \ll 1 \qquad (A.1.10)$$



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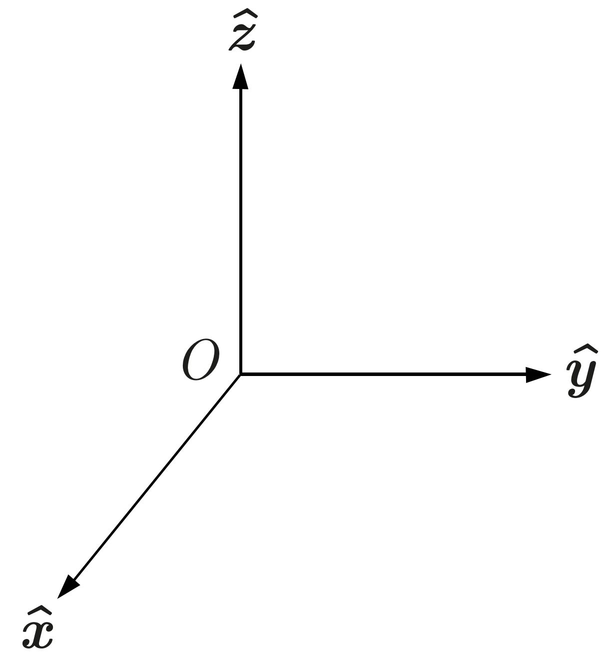
- Repère cartésien : $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$

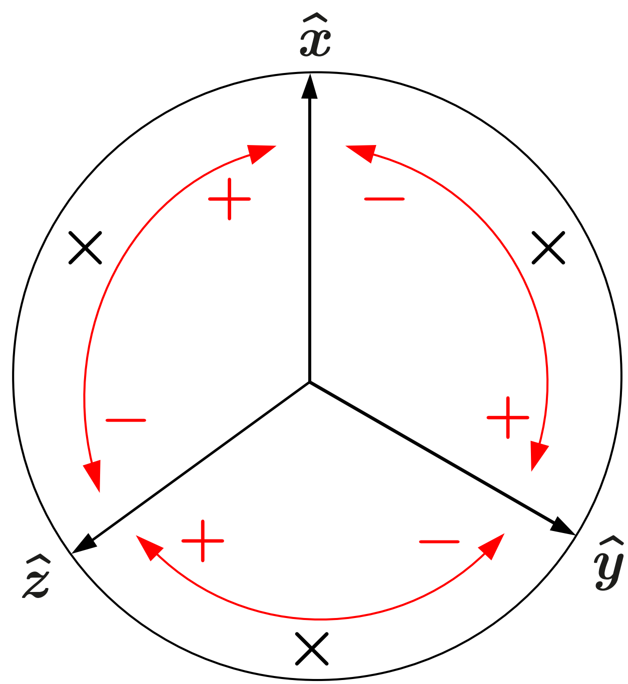


1. Règle de la main droite :

$$\textbf{index} \times \textbf{majeur} = \textbf{pouce} \qquad (A.1.11)$$

- $\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = +\,\hat{\boldsymbol{z}}$ et $\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}} = -\,\hat{\boldsymbol{z}}$
- $\hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = +\,\hat{\boldsymbol{x}}$ et $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{y}} = -\,\hat{\boldsymbol{x}}$
- $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}} = +\,\hat{\boldsymbol{y}}$ et $\hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}} = -\,\hat{\boldsymbol{y}}$

2. Règle du logo "Mercedes" :



1. $\circlearrowright \;\Rightarrow\; +$
2. $\circlearrowleft \;\Rightarrow\; -$

- **Dérivée temporelle du produit** $f * g$:
  1. $f \equiv f(t)$ et $g \equiv g(t)$: fonctions scalaires ou vectorielles du temps.
  2. $(*)$: produit algébrique, scalaire $(\cdot)$ ou vectoriel $(\times)$.

- **Variation du produit** $f * g$ **entre** t **et** $t + \Delta t$:

$$\begin{aligned}\Delta(f * g) &= f(t+\Delta t) * g(t+\Delta t) - f(t) * g(t) \\ &= f(t+\Delta t) * g(t+\Delta t) - f(t) * g(t+\Delta t) \\ &\quad + f(t) * g(t+\Delta t) - f(t) * g(t) \\ &= \Delta f * g(t+\Delta t) + f(t) * \Delta g\end{aligned} \tag{1.31}$$

- **Dérivée temporelle du produit** $f * g$:

$$\frac{d}{dt}(f * g) = \lim_{\Delta t \to 0} \frac{\Delta(f * g)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} * \underbrace{g(t+\Delta t)}_{\to g(t)} + \lim_{\Delta t \to 0} f(t) * \frac{\Delta g}{\Delta t}$$

$$\boxed{\frac{d}{dt}(f * g) = \frac{df}{dt} * g + f * \frac{dg}{dt}} \tag{1.32}$$